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MODIFICATION OF LINEAR FUNCTIONALS WITH DIRAC MASSES: CLASS OF THE MODIFIED LINEAR FUNCTIONAL

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To Professor Jairo A. Charris in memoriam

Abstract. In this paper we analyze modifications of some classical linear functionals. More precisely, we deal with a generalized Jacobi linear functional [2] and we find the class of this linear functional. Finally, a similar analysis for generalized Laguerre linear functional is carried out.

1. Introduction

In the last twenty years many authors have focused their research interest in the analysis of sequences of polynomials orthogonal with respect to modifications of classical linear functionals consisting in the addition of Dirac functionals or their derivatives at the end points of the orthogonality interval. The pioneering paper by T.H. Koornwinder [8] opened an important way to explore such families from several points of view: hypergeometric character, holonomic equations that these polynomials satisfy, electrostatic interpretation of their zeros, asymptotic properties, etc.

More recently, several authors [4, 5, 6] have considered modifications of classical weight functions via the addition of both Dirac functionals and their derivatives in the framework of bispectral Darboux transformations and WKB method for second order linear differential equations.

The aim of this contribution is twofold. First, we give a summary of the background on linear functionals which is needed in the theory of orthogonal polynomials. In particular, we introduce the Stieltjes function associated with a linear functional as well as the connection between modified linear functionals and the corresponding Stieltjes functions. Second, for semiclassical linear functionals we introduce the notion of class and establish the classification of modified linear functionals in terms of its class when Dirac functionals and their

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derivatives are added. We illustrate such a question for Jacobi and Laguerre linear functionals as examples of bounded and unbounded cases, respectively.

The structure of the paper is the following. In section 2, we introduce several modifications of linear functionals and study the relations for their corresponding Stieltjes functions. The section 3 deals with the semiclassical linear functionals and the first order linear differential equation that the Stieltjes function satisfies. In section 4, the classification of the modified Jacobi linear functionals introduced in [2] is given in terms of the Dirac masses. This section contains new results and gives the answer to a remark by P. Maroni concerning a result in [2]. Furthermore, a similar analysis for the Laguerre case is done.

2. Basic definitions

In this section, we introduce the concept of moment functionals. We study some properties of these functionals as well as the corresponding Stieltjes functions.

Let $\mathbb{E}$ be a real linear space. The map $\mathcal{L} : \mathbb{E} \mapsto \mathbb{R}$ is said to be a functional. This functional is additive if

$$\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y), \quad \forall x, y \in \mathbb{E},$$

and homogeneous if

$$\mathcal{L}(\alpha x) = \alpha \mathcal{L}(x), \quad \forall \alpha \in \mathbb{R}, \quad \forall x \in \mathbb{E}.$$

An additive and homogeneous functional is called linear functional.

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients and denote $\mathbb{P}_n$ the linear subspace of $\mathbb{P}$ of polynomials with real coefficients of degree at most n. Let us define the algebraic dual space of $\mathbb{P}$, denoted by $\mathbb{P}^*$, as

$$\mathbb{P}^* = \{\mathcal{U} : \mathbb{P} \to \mathbb{R}, \quad \text{such that } \mathcal{U} \text{ is a linear functional}\}.$$

The linear space $\mathbb{P}$ can be represented as

$$\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n, \quad \text{where } \mathbb{P}_n \subset \mathbb{P}_{n+1}.$$

Notice that $\mathbb{P}_n$ is a normed space. Indeed, the norm

$$\| p \|_n = \sum_{k=0}^{n} \frac{| p^{(k)}(0) |}{k!}, \quad \forall p \in \mathbb{P}_n,$$

induces a topology on $\mathbb{P}$. Therefore, $\mathbb{P}_n$ can be treated as a topological linear space. Considering the sequence of topological linear spaces $\{\mathbb{P}_n, \| \cdot \|_n\}_{n\geq 0}$ and taking inductive limit one gets a topology in the limit space $\mathbb{P}$.

From now on, we denote the action of a linear functional $\mathcal{U}$ on any polynomial $p \in \mathbb{P}$ as $\langle \mathcal{U}, p\rangle$. In particular

$$u_n = \langle \mathcal{U}, x^n \rangle, \quad n \geq 0,$$

is said to be the moment of order n of $\mathcal{U}$ and the sequence $\{u_n\}_{n\geq 0}$ is called the moment sequence associated with $\mathcal{U}$.

Definition 1. *Let $\{u_n\}_{n\geq 0}$ be a sequence on $\mathbb{R}$. $\mathcal{U}$ is said to be the moment functional associated with the sequence $\{u_n\}_{n\geq 0}$, if $\langle \mathcal{U}, x^n\rangle = u_n$ and it can be extended to $\mathbb{P}$ by linearity, i.e., if*

$$p(x) = \sum_{k=0}^{n} a_k x^k, \text{ then } \langle \mathcal{U}, p\rangle = \sum_{k=0}^{n} a_k u_k.$$

Definition 2. *$\mathcal{U}$ is said to be positive-definite if $\langle \mathcal{U}, q\rangle > 0$ for every non-identically zero polynomial $q(x)$ such that $q(x) \geq 0$, $\forall x \in \mathbb{R}$.*

The following theorem gives us a characterization of a positive-definite functional.

Theorem 3. *$\mathcal{U}$ is positive-definite if and only if its moments are all real and* $\det(u_{i+j})_{i,j=0}^{n} > 0$, $\forall n \in \mathbb{N}_0$.

Definition 4. *A linear functional is said to be quasi-definite if* $\det(u_{i+j})_{i,j=0}^{n}$ *does not vanish for all $n \geq 0$.*

Definition 5. *Let $\mathcal{U}$ be any moment functional. Define the Stieltjes function associated with $\mathcal{U}$ as the formal series*

$$\mathcal{S}_{\mathcal{U}}(z) = -\sum_{k=0}^{\infty} \frac{u_k}{z^{k+1}}.$$

In fact,

$$\mathcal{S}_{\mathcal{U}}(z) = -\sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \langle \mathcal{U}, x^n\rangle = -\sum_{n=0}^{\infty} \left\langle \mathcal{U}, \frac{x^n}{z^{n+1}} \right\rangle.$$

Notice that the geometric series $\sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} = \frac{1}{z-x}$ for $|x| < |z|$. Therefore, by convention, we consider

$$\sum_{n=0}^{\infty} \left\langle \mathcal{U}, \frac{x^n}{z^{n+1}} \right\rangle = \left\langle \mathcal{U}, \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} \right\rangle,$$

because of the uniform convergence of the series on a neighborhood of the infinity. Thus,

$$\mathcal{S}_{\mathcal{U}}(z) = -\left\langle \mathcal{U}_x, \frac{1}{z-x} \right\rangle = \left\langle \mathcal{U}_x, \frac{1}{x-z} \right\rangle,$$

where $\mathcal{U}_x$ means that $\mathcal{U}$ is acting on the variable x.

Now, we focus our attention on the Stieltjes function associated with modified moment functionals. Specifically, two modifications are considered. The first one consisting in the addition of Dirac linear functionals while the second one in the product by polynomials and rational functions.

Definition 6. *Let $\mathcal{U}$ be a moment functional. The derivative of the moment functional $\mathcal{U}$, denoted by $D\mathcal{U}$, is defined as*

$$\langle D\mathcal{U}, p\rangle = -\langle \mathcal{U}, p'\rangle, \quad p \in \mathbb{P},$$

where D denotes the usual distributional derivative.

Clearly, from the previous definition the k-th derivative of the moment functional $\mathcal{U}$

$$\left\langle D^{(k)}\mathcal{U}, p\right\rangle = (-1)^k \left\langle \mathcal{U}, p^{(k)}\right\rangle, \quad k \geq 0,$$

yields.

Let $a \in \mathbb{R}$. The action of the Dirac linear functional supported at a, $\delta(x-a)$, is defined as

$$\langle \delta(x-a), p\rangle = p(a), \quad \forall p \in \mathbb{P}.$$

In particular, the moment sequence of a Dirac functional supported at a is given by

$$\langle \delta(x-a), x^n\rangle = \begin{cases} a^n, & n > 0, \\ 1, & n = 0, \end{cases}$$

and its Stieltjes function is

$$\mathcal{S}(z) = -\sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}} = \frac{1}{a-z}, \quad |a| < |z|. \tag{1}$$

Taking derivatives in both hand sides of (1) one gets

$$\sum_{n=0}^{\infty}(n+1)\frac{a^n}{z^{n+2}} = \frac{1}{(z-a)^2}. \tag{2}$$

Now, an iterative derivation of (2) leads to the following expression

$$\sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+i)\frac{a^n}{z^{n+i+1}} = \frac{i!}{(z-a)^{i+1}}, \quad i \geq 1.$$

2.1. **Addition of Dirac functionals.** Let $a \in \mathbb{R}, \mathcal{U}$ a moment functional, and $\{u_n\}_{n\geq 0}$ be its moment sequence. Let us consider a new moment functional $\widetilde{\mathcal{U}}$ obtained as the modification of $\mathcal{U}$ by the addition of the derivatives (up to order $N \in \mathbb{N}$) of the Dirac functional supported at a, i.e.,

$$\widetilde{\mathcal{U}} = \mathcal{U} + \sum_{i=0}^{N} M_i \delta^{(i)}(x-a), \tag{3}$$

where $M_i \in \mathbb{R}$ $(i = 0, 1, \ldots, N)$ and $\delta^{(i)}(x-a)$ denotes the i-th derivative of the Dirac functional supported at a. Furthermore,

$$\begin{aligned} u_n^{(i)} &\stackrel{\text{def}}{=} \left\langle \delta^{(i)}(x-a), x^n\right\rangle = (-1)^i \left\langle \delta(x-a), D^{(i)}(x^n)\right\rangle \\ &= (-1)^i n(n-1)\cdots(n-i+1)a^{n-i}, \end{aligned} \tag{4}$$

and the Stieltjes function associated with $\delta^{(i)}$ is

$$\mathcal{S}_{\delta^{(i)}}(z) = (-1)^{i+1} i! \frac{1}{(z-a)^{i+1}} = D^{(i)}\left(\frac{1}{a-z}\right). \tag{5}$$

Now, we express the moments of $\widetilde{\mathcal{U}}$ in terms of the moments of $\mathcal{U}$,

$$\begin{aligned}
\tilde{u}_n = \left\langle \widetilde{\mathcal{U}}, x^n \right\rangle &= \left\langle \mathcal{U} + \sum_{i=0}^{N} M_i \delta^{(i)}(x-a), x^n \right\rangle \\
&= \langle \mathcal{U}, x^n \rangle + \sum_{i=0}^{N} M_i \left\langle \delta^{(i)}(x-a), x^n \right\rangle \\
&= u_n + \sum_{i=0}^{N} M_i (-1)^i n(n-1)\cdots(n-i+1) a^{n-i}.
\end{aligned}$$

On the other hand, from (4) the following relation holds

$$\tilde{u}_n = u_n + \sum_{i=0}^{N} M_i u_n^{(i)}.$$

For our later purpose (see Theorem 19) it is very useful to express the Stieltjes function of $\widetilde{\mathcal{U}}$ in terms of $\mathcal{S}_{\mathcal{U}}(z)$, so

$$\begin{aligned}
\mathcal{S}_{\widetilde{\mathcal{U}}}(z) = -\sum_{n=0}^{\infty} \frac{\tilde{u}_n}{z^{n+1}} &= -\sum_{n=0}^{\infty} \frac{u_n + \sum_{i=1}^{N} M_i u_n^{(i)}}{z^{n+1}} \\
&= -\sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}} - \sum_{n=0}^{\infty} \sum_{i=0}^{N} M_i \frac{u_n^{(i)}}{z^{n+1}} \\
&= \mathcal{S}_{\mathcal{U}}(z) + \sum_{i=0}^{N} M_i \mathcal{S}_{\delta^{(i)}}(z).
\end{aligned}$$

Taking into account (5)

$$\begin{aligned}
\mathcal{S}_{\widetilde{\mathcal{U}}}(z) &= \mathcal{S}_{\mathcal{U}}(z) + \sum_{i=0}^{N} M_i D^{(i)}\left(\frac{1}{a-z}\right) = \mathcal{S}_{\mathcal{U}}(z) + \sum_{i=0}^{N} M_i \frac{(-1)^{i+1} i!}{(z-a)^{i+1}} \\
&= \mathcal{S}_{\mathcal{U}}(z) + \frac{\sum_{i=0}^{N} (-1)^{i+1} i! M_i (z-a)^{N-i}}{(z-a)^{N+1}} \\
&= \mathcal{S}_{\mathcal{U}}(z) + \frac{\sum_{i=0}^{N} (-1)^{i+1} i! M_i (z-a)^{N-i}}{(z-a)^{N+1}} = \mathcal{S}_{\mathcal{U}}(z) + \frac{R_N(z)}{(z-a)^{N+1}},
\end{aligned}$$

where $R_N(z)$ denotes the polynomial

$$R_N(z) = \sum_{i=0}^{N} (-1)^{i+1} i! M_i (z-a)^{N-i}.$$

Notice that in order to add a new derivative of the Dirac functional in (3) the corresponding Stieltjes function can be easily computed taking into account that the polynomials $R_N(z)$ can be defined recursively

$$(z-a)R_{N+1}(z) = (z-a)R_N(z) + (-1)^N (N+1)! M_{N+1}.$$

2.2. **Left-multiplication by a polynomial.**

Definition 7. *Let $q \in \mathbb{P}$. The linear functional $\widetilde{\mathcal{U}} = q\mathcal{U}$ is said to be the left-multiplication of $\mathcal{U}$ by a polynomial q if*

$$\left\langle \widetilde{\mathcal{U}}, p \right\rangle = \langle \mathcal{U}, qp \rangle, \quad p \in \mathbb{P}. \tag{6}$$

We consider

$$q(x) = \sum_{i=0}^{k} q_i x^i, \quad q_k \neq 0.$$

Thus

$$\langle q(x)\mathcal{U}, x^n \rangle = \langle \mathcal{U}, q(x)x^n \rangle = \sum_{i=0}^{k} q_i u_{i+n}, \quad n \geq 0. \tag{7}$$

As a consequence, the Stieltjes function of $\widetilde{\mathcal{U}}$ is

$$\begin{aligned}
\mathcal{S}_{\widetilde{\mathcal{U}}}(z) &= -\sum_{n=0}^{\infty} \frac{\tilde{u}_n}{z^{n+1}} = -\sum_{n=0}^{\infty} \frac{\sum_{i=0}^{k} q_i u_{i+n}}{z^{n+1}} \\
&= -\sum_{i=0}^{k} q_i \sum_{n=0}^{\infty} \frac{u_{i+n}}{z^{n+1}} = -\sum_{i=0}^{k} q_i z^i \sum_{n=0}^{\infty} \frac{u_{i+n}}{z^{i+n+1}}.
\end{aligned}$$

Let $p_i(x) = u_0 + u_1 x + \ldots + u_i x^i$, $i = 0, 1, ...$, and consider, by convention, $p_{-1}(x) = 0$. From this definition, we deduce that the polynomial $p_i(x)$ satisfies the recurrence formula $p_{i+1}(x) = p_i(x) + u_{i+1}x^{i+1}$, $i \geq 0$. Then

$$\begin{aligned}
\mathcal{S}_{\widetilde{\mathcal{U}}}(z) &= \sum_{i=0}^{k} q_i z^i \left[\mathcal{S}_{\mathcal{U}}(z) + \frac{1}{z} p_{i-1}(z^{-1}) \right] \\
&= q(z)\mathcal{S}_{\mathcal{U}}(z) + \sum_{i=0}^{k} q_i z^{i-1} p_{i-1}(z^{-1}) \\
&= q(z)\mathcal{S}_{\mathcal{U}}(z) + \sum_{i=1}^{k} q_i p_{i-1}^{*}(z),
\end{aligned}$$

where $p_i^*(z)$ is the reciprocal polynomial of $p_i(z)$, i.e,

$$p_i^*(z) = \sum_{j=0}^{i} u_j z^{i-j}. \tag{8}$$

Notice that the polynomial $p_i^*(z)$ satisfies also a recurrence relation, $p_{i+1}^*(z) = zp_i^*(z) + u_{i+1}$, $i \geq 0$. Therefore,

$$\mathcal{S}_{\widetilde{\mathcal{U}}}(z) = q(z)\mathcal{S}_{\mathcal{U}}(z) + \sum_{i=1}^{k} q_i p_{i-1}^*(z). \tag{9}$$

Definition 8. *Let $h \in \mathbb{P}$ be a polynomial of degree n*

$$h(x) = \sum_{i=0}^{n} h_i x^i.$$

The polynomial

$$(\mathcal{U}h)(z) = \begin{pmatrix} 1, & z, & \ldots, & z^n \end{pmatrix} \begin{pmatrix} u_0 & u_1 & \ldots & u_n \\ 0 & u_0 & \ldots & u_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{pmatrix}, \tag{10}$$

is said to be the right-multiplication of the linear functional $\mathcal{U}$ by h.

Definition 9. *Let $a \in \mathbb{R}$ and $\Theta_a(p) = \dfrac{p(z) - p(a)}{z - a}$ where $p \in \mathbb{P}$. The linear functional, $(z-a)^{-1}\mathcal{U}$, the left-multiplication of $\mathcal{U}$ by the rational function $(z-a)^{-1}$, is defined as*

$$\left\langle (z-a)^{-1}\mathcal{U}, p \right\rangle = \langle \mathcal{U}, \Theta_a(p) \rangle.$$

Lemma 10. *Let $q \in \mathbb{P}$ be such that*

$$q(x) = \sum_{i=0}^{n} q_i x^i, \quad q_n \neq 0.$$

Then

$$\big(\mathcal{U}\Theta_0(q)\big)(x) = \sum_{i=1}^{n} q_i p_{i-1}^*(x), \tag{11}$$

where $p_i^*(x)$ is the reciprocal polynomial of $p_i(x)$ defined in (8).

Proof. From (11)

$$\begin{aligned}\sum_{i=1}^{n} q_i p_{i-1}^*(x) &= q_1 u_0 + q_2(u_0 x + u_1) + \cdots + q_k(u_0 x^{k-1} + \cdots + u_{k-1}) \\ &= (q_1 u_0 + \cdots + q_k u_{k-1}) + (q_2 u_0 + \cdots + q_k u_{k-2})x + \cdots + q_k u_0 x^{k-1}.\end{aligned}$$

Equivalently,

$$\begin{aligned}\sum_{i=1}^{n} q_i p^*_{i-1}(x) &= \begin{pmatrix} 1, & x, & \ldots, & x^{k-1} \end{pmatrix} \begin{pmatrix} u_0 & u_1 & \ldots & u_{k-1} \\ 0 & u_0 & \ldots & u_{k-2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & u_0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{pmatrix} \\ &= (\mathcal{U}\Theta_0(q))\,(x),\end{aligned}$$

and our statement holds. □

By lemma 10 we get

$$\mathcal{S}_{\widetilde{\mathcal{U}}}(z) = q(z)\mathcal{S}_{\mathcal{U}}(z) + \big(\mathcal{U}\Theta_0(q)\big)(z). \tag{12}$$

The following lemma (see [1]) will be very useful in the sequel.

Lemma 11. *For every polynomial q and $a \in \mathbb{R}$,*

$$\langle \mathcal{U}, \Theta_a(q)\rangle = \big(\mathcal{U}\Theta_0(q)\big)(a).$$

Remark 12. *Definition 8 provides us the following expression of the polynomial*

$$(\mathcal{U}h)\,(z) = \sum_{i=0}^{k}\left(\sum_{j=0}^{k-i} u_j q_{j+1}\right) z^i = \sum_{i=0}^{k}\left(\sum_{j=0}^{i} \frac{u_j}{z^j}\right) q_i z^i.$$

2.3. **Inverse linear functional.** In this section, we introduce the inverse linear functional for a linear functional $\mathcal{U}$ provided that it exists. For an inverse linear functional we mean that there exists a linear functional $\mathcal{U}^{-1}$ such that $\mathcal{U}\mathcal{U}^{-1} = \delta(x)$, i.e, the product of both functionals is the Dirac functional supported at the origin. This means that its moment of order zero is exactly one and the other moments are identically zero. For that reason, we need to define the product of functionals.

Definition 13. *For any $\mathcal{U}, \mathcal{V} \in \mathbb{P}^*$, the product of both moment functionals $\mathcal{U}\mathcal{V}$ is a moment functional on $\mathbb{P}$ such that*

$$\langle \mathcal{U}\mathcal{V}, p\rangle = \langle \mathcal{U}, \mathcal{V}p\rangle\,, \quad \textit{for all} \quad p \in \mathbb{P}.$$

The corresponding moments are

$$(uv)_n = \sum_{k=0}^{n} u_k v_{n-k}, \quad n \geq 0, \tag{13}$$

where $\{u_k\}_{k=0}^{\infty}$ and $\{v_k\}_{k=0}^{\infty}$ are the moment sequences of $\mathcal{U}$ and $\mathcal{V}$, respectively.

Let us consider the moment functional $\mathcal{U}$ as well as its moment sequence $\{u_k\}_{k=0}^{\infty}$. In order to guarantee the existence of the inverse linear functional $\mathcal{U}^{-1}$, the moment of order zero must be equal to one ($u_0 v_0 = 1$). Thus, u_0 cannot be zero, and obviously from the above we deduce that $v_0 \neq 0$. Let us denote $\mathcal{V} = \mathcal{U}^{-1}$ and $\{v_k\}_{k=0}^{\infty}$ the moment sequence associated with $\mathcal{V}$. From (13)

$$(uv)_n = u_0 v_n + \sum_{k=1}^{n} u_k v_{n-k} = 0, \quad n \geq 1.$$

Taking into account $u_0 \neq 0$ the moment of order n associated with the moment functional $\mathcal{V}$ is

$$v_n = -\frac{1}{u_0} \sum_{k=1}^{n} u_k v_{n-k}, \quad n \geq 1. \tag{14}$$

Thus, the inverse linear functional is defined. Indeed, since $u_0 \neq 0$ the existence of such a functional is always guaranteed by the sequence of moments.

On the other hand, for every linear functional $\mathcal{U}$ such that there exists $\mathcal{U}^{-1}$, i.e $\mathcal{U}\mathcal{U}^{-1} = \delta(x)$ by using the moments of $\mathcal{U}\mathcal{U}^{-1}$, denoted by w_k ($w_0 = u_0 v_0 = 1$) one deduces that u_0 as well as v_0 are not identically zero.

Theorem 14. *Let $\mathcal{U}$ be a moment functional. The necessary and sufficient condition for the existence of $\mathcal{U}^{-1}$ is $u_0 \neq 0$ (the moment of order zero of $\mathcal{U}$ does not vanish).*

Once the moments of the functional $\mathcal{V}$ are known, the Stieltjes function associated with the functional $\mathcal{V}$ can be given in terms of the Stieltjes function associated with $\mathcal{U}$. Indeed,

$$\begin{aligned}
\mathcal{S}_{\mathcal{U}}(z)\mathcal{S}_{\mathcal{V}}(z) &= \left(\sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}}\right)\left(\sum_{j=0}^{\infty} \frac{v_j}{z^{j+1}}\right) = \sum_{n,j=0}^{\infty} \frac{u_n v_j}{z^{n+j+2}} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} u_k v_{n-k}}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(uv)_n}{z^{n+1}} = \frac{1}{z^2},
\end{aligned}$$

or equivalently,

$$z^2 \mathcal{S}_{\mathcal{U}}(z)\mathcal{S}_{\mathcal{V}}(z) = 1.$$

3. Semiclassical orthogonal polynomials

It is very well known, see [9], that classical orthogonal polynomials, $p_n(x)$, satisfy the Rodrigues formula

$$p_n(x) = \frac{k_n}{w(x)} \frac{d^n}{dx^n}\Big(\sigma^n(x) w(x)\Big),$$

where k_n is a real constant, $\sigma(x) \in \mathbb{P}_2$ and $w(x)$ is the weight function supported on the ortogonality interval. As a natural generalization, polynomial sequences orthogonal with respect to a moment functional $\mathcal{U}$ satisfying a Pearson equation

$$D(\phi\mathcal{U}) = \psi\mathcal{U},$$

with $\deg \phi = k \geq 0$ and $\deg \psi = l \geq 1$ can be considered. Such polynomial sequences are said to be semiclassical orthogonal polynomial sequences. The moment functional $\mathcal{U}$ is called a semiclassical functional.

Definition 15. *If (ϕ, ψ) are polynomials of minimum degree such that*

$$D(\phi\mathcal{U}) = \psi\mathcal{U},$$

define the class of $\mathcal{U}$ as the nonnegative integer number s such that

$$s = \max\{\deg \phi - 2, \deg \psi - 1\}.$$

Theorem 16. *Let $\mathcal{U}$ be a quasi-definite moment functional such that*

$$D(\phi\mathcal{U}) = \psi\mathcal{U}, \tag{15}$$

where $\phi, \psi \in \mathbb{P}$ with $\deg \phi = k \geq 0$ and $\deg \psi = l \geq 1$. Then, the Stieltjes function $\mathcal{S}_{\mathcal{U}}(z)$ satisfies

$$\phi(z)\mathcal{S}'_{\mathcal{U}}(z) = B(z)\mathcal{S}_{\mathcal{U}}(z) + C(z), \tag{16}$$

where $B(z)$ and $C(z)$ are given in terms of ϕ and ψ by

$$\begin{aligned} B(z) &= -\phi'(z) + \psi(z), \\ C(z) &= -\left(\mathcal{U}\Theta_0(\phi)\right)'(z) + \left(\mathcal{U}\Theta_0(\psi)\right)(z). \end{aligned}$$

Proof. From the Pearson equation (15)

$$\left\langle D(\phi\mathcal{U}), \frac{1}{x-z} \right\rangle = \left\langle \psi\mathcal{U}, \frac{1}{x-z} \right\rangle. \tag{17}$$

Let us handel the left hand side of the above expression

$$\begin{aligned} \left\langle D(\phi\mathcal{U}), \frac{1}{x-z} \right\rangle &= \left\langle \phi\mathcal{U}, \frac{1}{(x-z)^2} \right\rangle = \left\langle \mathcal{U}, \frac{\phi(x)}{(x-z)^2} \right\rangle \\ &= \left\langle \mathcal{U}, \frac{\phi(x)-\phi(z)}{(x-z)^2} \right\rangle + \phi(z) \left\langle \mathcal{U}, \frac{1}{(x-z)^2} \right\rangle \\ &= \phi(z)\mathcal{S}'_{\mathcal{U}}(z) + \left\langle \mathcal{U}, \frac{\phi(x)-\phi(z)-\phi'(z)(x-z)}{(x-z)^2} \right\rangle \\ &+ \phi'(z) \left\langle \mathcal{U}, \frac{1}{x-z} \right\rangle. \end{aligned}$$

Therefore,

$$\left\langle D(\phi\mathcal{U}), \frac{1}{x-z} \right\rangle = \phi(z)\mathcal{S}'_{\mathcal{U}}(z) + \phi'(z)\mathcal{S}_{\mathcal{U}}(z) + \left\langle (x-z)^{-2}\mathcal{U}, \phi(x) \right\rangle. \tag{18}$$

On the other hand, from the right hand side in (17) and from the Stieltjes function of the moment functional associated with the left product by a polynomial one gets

$$\left\langle \psi\mathcal{U}, \frac{1}{x-z} \right\rangle = \psi(z)\mathcal{S}_{\mathcal{U}}(z) + \big(\mathcal{U}\Theta_0(\psi)\big)(z). \tag{19}$$

So, from (18) and (19) we deduce the result

$$\phi(z)\mathcal{S}'_u(z)+\underbrace{\big(\phi'(z)-\psi(z)\big)}_{B(z)}\mathcal{S}_{\mathcal{U}}(z)+\underbrace{\big\langle(x-z)^{-2}\mathcal{U},\phi\big\rangle-\big(\mathcal{U}\Theta_0(\psi)\big)(z)}_{C(z)}=0. \quad (20)$$

□

Notice that

$$\big\langle(x-z)^{-2}\mathcal{U},\phi\big\rangle=\left\langle\mathcal{U},\frac{\phi(x)-\phi(z)-\phi'(z)(x-z)}{(x-z)^2}\right\rangle,$$

and, furthermore,

$$\begin{aligned}\frac{d}{dz}\left\langle(x-z)^{-1}\mathcal{U},\phi\right\rangle &= \left\langle\mathcal{U},\frac{\phi(x)-\phi(z)}{(x-z)^2}-\frac{\phi'(z)}{x-z}\right\rangle\\ &= \left\langle\mathcal{U},\frac{\phi(x)-\phi(z)-\phi'(z)(x-z)}{(x-z)^2}\right\rangle.\end{aligned}$$

From the both previous expressions and using lemma 11, we get

$$\big\langle(x-z)^{-2}\mathcal{U},\phi\big\rangle=\frac{d}{dz}\big(\mathcal{U}\Theta_0(\phi)\big)(z).$$

Therefore, if

$$\begin{aligned}F(z) &= \big\langle(x-z)^{-1}\mathcal{U},\phi\big\rangle=\left(\mathcal{U}\Theta_0(\phi)\right)(z),\\ G(z) &= \big\langle(x-z)^{-1}\mathcal{U},\psi\big\rangle=\left(\mathcal{U}\Theta_0(\psi)\right)(z),\end{aligned}$$

then the Stieltjes function $\mathcal{S}_{\mathcal{U}}(z)$ satisfies the following differential equation

$$\phi(z)\mathcal{S}'_{\mathcal{U}}(z)+\big(\phi'(z)-\psi(z)\big)\mathcal{S}_{\mathcal{U}}(z)+F'(z)-G(z)=0. \quad (21)$$

Remark 17. *Previous theorem 16 plays a crucial role in the computation of the class of the semiclassical functionals given by a perturbation via the addition of Dirac masses (see section 4 below). In fact, the expressions (20) and (21) will be useful.*

The left-multiplication of any rational function by a moment functional $\mathcal{U}$ acting on any polynomial q can be expressed in terms of the moments of the functional $\mathcal{U}$. Then

$$\begin{aligned}\big\langle(x-z)^{-1}\mathcal{U},q\big\rangle &= \left\langle\mathcal{U},q'(z)+\frac{q''(z)}{2}(x-z)+\cdots+\frac{q^{(k)}(z)}{k!}(x-z)^{k-1}\right\rangle\\ &= q'(z)\left\langle\mathcal{U},1\right\rangle+\frac{q''(z)}{2}\left\langle\mathcal{U},x-z\right\rangle+\cdots+\frac{q^{(k)}(z)}{k!}\left\langle\mathcal{U},(x-z)^{k-1}\right\rangle.\end{aligned}$$

In the same way

$$\begin{aligned}\big\langle(x-z)^{-2}\mathcal{U},q\big\rangle &= \left\langle\mathcal{U},\frac{q''(z)}{2}+\frac{q'''(z)}{3!}(x-z)+\cdots+\frac{q^{(k)}(z)}{k!}(x-z)^{k-2}\right\rangle\\ &= \frac{q''(z)}{2}\left\langle\mathcal{U},1\right\rangle+\frac{q'''(z)}{3!}\left\langle\mathcal{U},x-z\right\rangle+\cdots+\frac{q^{(k)}(z)}{k!}\left\langle\mathcal{U},(x-z)^{k-2}\right\rangle,\end{aligned}$$

and so on. In particular

$$\big\langle [p(x)]^{-1}\mathcal{U}, q(x)\big\rangle = \left\langle \mathcal{U}, \frac{q(x)-(L_p q)(x)}{p(x)}\right\rangle,$$

where $(L_p q)(x)$ denotes the interpolatory polynomial of q at the zeros of p taking into account their multiplicity.

Remark 18. *If the polynomial coefficients ϕ, B, and C in (16) are co-prime, i.e., their greatest common divisor is* 1*, the order of the class is*

$$s = \max\{\deg\phi - 2, \deg(B+\phi') - 1\}.$$

Otherwise, the differential equation (16) can be reduced then the order of the class is smaller since the degree of the polynomial coefficients is less.

4. Modification of linear functionals

We consider the classical Jacobi functional

$$\langle \mathcal{J}_{\alpha,\beta}, p\rangle = \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}p(x)dx, \quad p\in\mathbb{P}, \quad \alpha,\beta\in(-1,+\infty).$$

We deal with a perturbation deduced by the addition of two Dirac linear functionals as well as their derivatives at the points $x=-1$ and $x=1$. The sequences of orthogonal polynomials associated with this kind of perturbations has been analyzed in [2]. More precisely,

$$\widetilde{\mathcal{J}}_{\alpha,\beta} = \mathcal{J}_{\alpha,\beta} + M_1\delta(x-1) + M_2\delta(x+1) + M_3\delta'(x-1) + M_4\delta'(x+1). \tag{22}$$

It is known, see [2], that the functional $\mathcal{J}_{\alpha,\beta}$ satisfies the Pearson equation

$$D\left[(1-x^2)\mathcal{J}_{\alpha,\beta}\right] = \left[\beta-\alpha-(\alpha+\beta+2)x\right]\mathcal{J}_{\alpha,\beta}.$$

From this equation we can compute the Pearson equation satisfied by the new perturbed functional, assuming that $M_3M_4\neq 0$,

$$D\left[(1-x^2)^3\widetilde{\mathcal{J}}_{\alpha,\beta}\right] = \left[\beta-\alpha-(\alpha+\beta+6)x\right]\left(1-x^2\right)^2\widetilde{\mathcal{J}}_{\alpha,\beta}.$$

From now on, we denote

$$\begin{aligned} \phi(x) &= (1-x^2)^3,\\ \psi(x) &= \big(\beta-\alpha-(\alpha+\beta+6)x\big)(1-x^2)^2. \end{aligned}$$

The Stieltjes function associated with the Jacobi linear functional $\mathcal{J}_{\alpha,\beta}$ satisfies a differential equation

$$(1-x^2)\mathcal{S}'(x) = (\beta-\alpha-(\alpha+\beta)x)\mathcal{S}(x) - (\alpha+\beta+1)u_0, \tag{23}$$

where u_0 is the moment of order zero of the Jacobi functional, which we assume normalized.

On the other hand, the Stieltjes function $\widetilde{\mathcal{S}}$ associated with the moment functional $\widetilde{\mathcal{J}}_{\alpha,\beta}$ is

$$\widetilde{\mathcal{S}}(x) = \mathcal{S}(x) - \frac{M_1}{x-1} - \frac{M_2}{x+1} + \frac{M_3}{(x-1)^2} + \frac{M_4}{(x+1)^2}.$$

Therefore,

$$\widetilde{\mathcal{S}}'(x) = \mathcal{S}'(x) + \frac{M_1}{(x-1)^2} + \frac{M_2}{(x+1)^2} - \frac{2M_3}{(x-1)^3} - \frac{2M_4}{(x+1)^3},$$

that is

$$\mathcal{S}'(x) = \widetilde{\mathcal{S}}'(x) - \frac{M_1}{(x-1)^2} - \frac{M_2}{(x+1)^2} + \frac{2M_3}{(x-1)^3} + \frac{2M_4}{(x+1)^3}.$$

Substituting in (23)

$$\begin{gathered}(1-x^2)\widetilde{\mathcal{S}}'(x) - (1-x^2)\left[\frac{M_1}{(x-1)^2} + \frac{M_2}{(x+1)^2} - \frac{2M_3}{(x-1)^3} - \frac{2M_4}{(x+1)^3}\right] \\ = [\beta-\alpha-(\alpha+\beta)x]\left[\widetilde{\mathcal{S}}(x) + \frac{M_1}{x-1} + \frac{M_2}{x+1} - \frac{M_3}{(x-1)^2} - \frac{M_4}{(x+1)^2}\right] \\ -(\alpha+\beta+1)u_0.\end{gathered}$$

So

$$\begin{gathered}(1-x^2)\widetilde{\mathcal{S}}'(x) = \big(\beta-\alpha-(\alpha+\beta)x\big)\widetilde{\mathcal{S}}(x) \\ +[\beta-\alpha-(\alpha+\beta)x]\left[\frac{M_1}{x-1} + \frac{M_2}{x+1} - \frac{M_3}{(x-1)^2} - \frac{M_4}{(x+1)^2}\right] \\ +(1-x^2)\left[\frac{M_1}{(x-1)^2} + \frac{M_2}{(x+1)^2} - \frac{2M_3}{(x-1)^3} - \frac{2M_4}{(x+1)^3}\right] - (\alpha+\beta+1)u_0,\end{gathered}$$

i.e

$$\begin{gathered}(1-x^2)\widetilde{\mathcal{S}}'(x) = [\beta-\alpha-(\alpha+\beta)x]\,\widetilde{\mathcal{S}}(x) \\ +[\beta-\alpha-(\alpha+\beta)x]\left[\frac{M_1(x+1)+M_2(x-1)}{x^2-1} - \frac{M_3(x+1)^2+M_4(x-1)^2}{(x^2-1)^2}\right] \\ +(1-x^2)\left[\frac{M_1(x+1)^2+M_2(x-1)^2}{(x^2-1)^2} - 2\frac{M_3(x+1)^3+M_4(x-1)^3}{(x^2-1)^3}\right] \\ -(\alpha+\beta+1)u_0.\end{gathered}$$

Finally,

$$\begin{gathered}(1-x^2)\widetilde{\mathcal{S}}'(x) = \big[\beta-\alpha-(\alpha+\beta)x\big]\widetilde{\mathcal{S}}(x) \\ +\big[\beta-\alpha-(\alpha+\beta)x\big]\left[\frac{M_1(x+1)+M_2(x-1)}{x^2-1} - \frac{M_3(x+1)^2+M_4(x-1)^2}{(x^2-1)^2}\right] \\ +\left[\frac{M_1(x+1)^2+M_2(x-1)^2}{(x^2-1)} + 2\frac{M_3(x+1)^3+M_4(x-1)^3}{(x^2-1)^2}\right] - (\alpha+\beta+1)u_0,\end{gathered}$$

or equivalently,

$$
\begin{gathered}
(1-x^2)\widetilde{\mathcal{S}}'(x) = \big[\beta-\alpha-(\alpha+\beta)x\big]\widetilde{\mathcal{S}}(x) \\
-\big[\beta-\alpha-(\alpha+\beta)x\big]\left[\frac{(1-x^2)\left[M_1(x+1)+M_2(x-1)\right]}{(1-x^2)^2}\right. \\
\left. +\ \frac{M_3(x+1)^2+M_4(x-1)^2}{(1-x^2)^2}\right] \\
+\left[\frac{(x^2-1)\left[M_1(x+1)^2+M_2(x-1)^2\right]+2M_3(x+1)^3+2M_4(x-1)^3}{(1-x^2)^2}\right] \\
-(\alpha+\beta+1)u_0.
\end{gathered}
$$

The previous expression can be written as

$$
\begin{gathered}
(1-x^2)^3\widetilde{\mathcal{S}}'(x) = \big[\beta-\alpha-(\alpha+\beta)x\big]\big(1-x^2\big)^2\widetilde{\mathcal{S}}(x) \\
-\big[\beta-\alpha-(\alpha+\beta)x\big]\Big[(1-x^2)\big[M_1(x+1)+M_2(x-1)\big] \\
+\ M_3(x+1)^2+M_4(x-1)^2\Big] \\
+\big(x^2-1\big)\left[M_1(x+1)^2+M_2(x-1)^2\right]+2M_3(x+1)^3+2M_4(x-1)^3 \\
-(\alpha+\beta+1)(1-x^2)^2u_0.
\end{gathered}
$$

Thus,

$$
(1-x^2)^3\widetilde{\mathcal{S}}'(x) = \big[\beta-\alpha-(\alpha+\beta)x\big](1-x^2)^2\widetilde{\mathcal{S}}(x)+C(x),
$$

where

$$
\begin{gathered}
C(x) = \big[\beta-\alpha-(\alpha+\beta)x\big]\left[(x^2-1)\left[M_1(x+1)+M_2(x-1)\right]\right. \\
\left. -M_3(x+1)^2-M_4(x-1)^2\right] \\
+(x^2-1)\left[M_1(x+1)^2+M_2(x-1)^2\right]+2M_3(x+1)^3+2M_4(x-1)^3 \\
-(1-x^2)^2(\alpha+\beta+1)u_0.
\end{gathered}
$$

Taking derivatives

$$
\begin{gathered}
C'(x) = 4x(1-x^2)(\alpha+\beta+1)u_0 \\
-(\alpha+\beta)\left[(x^2-1)\left[M_1(x+1)+M_2(x-1)\right]-M_3(x+1)^2-M_4(x-1)^2\right] \\
+\left(\beta-\alpha-(\alpha+\beta)x\right)\left[2x\left[M_1(x+1)+M_2(x-1)\right]+(x^2-1)(M_1+M_2)\right. \\
\left. -2M_3(x+1)-2M_4(x-1)\right] \\
+2x\left[M_1(x+1)^2+M_2(x-1)^2\right]+(x^2-1)\left[2M_1(x+1)+2M_2(x-1)\right] \\
+6M_3(x+1)^2+6M_4(x-1)^2.
\end{gathered}
$$

Evaluating $C(x)$ at $x=1$ and $x=-1$, we get

$$
\begin{aligned}
C(1) &= 8(\alpha+2)M_3, \\
C(-1) &= -8(\beta+2)M_4.
\end{aligned}
$$

4.1. **Order of the class of the modified functional $\widetilde{\mathcal{J}}_{\alpha,\beta}$.** The Stieltjes function $\widetilde{\mathcal{S}}$ of the perturbed Jacobi functional is given in terms of $\mathcal{S}$ by

$$\widetilde{\mathcal{S}}(x) = -\sum_{n=0}^{\infty} \frac{\tilde{u}_n}{z^{n+1}} = \mathcal{S}(x) + M_1 \mathcal{S}_{\delta_1}(x) + M_2 \mathcal{S}_{\delta_{-1}}(x) + M_3 \mathcal{S}_{\delta_1^{(1)}}(x) + M_4 \mathcal{S}_{\delta_{-1}^{(1)}}(x),$$

where $\mathcal{S}_{\delta_a}(x)$ and $\mathcal{S}_{\delta_a^{(i)}}(x)$, $a \in \mathbb{R}$, denote the Stieltjes functions associated with the Dirac functionals $\delta(x-a)$ and $\delta^{(i)}(x-a)$, respectively. As we have seen in section 3, from theorem 16, this function satisfies the differential equation

$$\phi(x)\widetilde{\mathcal{S}}'(x) = B(x)\widetilde{\mathcal{S}}(x) + C(x).$$

From this equation, we will study the class of the modified functional. Let us denote s and $\tilde{s}$ the class of the functionals $\mathcal{J}_{\alpha,\beta}$ and $\widetilde{\mathcal{J}}_{\alpha,\beta}$, respectively. In order to reduce the above differential equation we have to study when the common zeros of ϕ and B are zeros of C as well. Several different cases are considered:

A If $C(1) \neq 0$ and $C(-1) \neq 0$, that is $M_3 M_4 \neq 0$, then the differential equation cannot be reduced because the polynomial coefficients are co-prime and the class of the functional $\tilde{J}_{\alpha,\beta}$ is

$$\tilde{s} = \max\{\deg \phi - 2, \deg \psi - 1\} = 4.$$

B If $M_3 = 0$ and $M_4 \neq 0$ the equation can be reduced because the polynomial coefficients have $x = 1$ as a common zero, so the equation can be divided by $1 - x$. Let us denote

$$\begin{aligned} \phi(x) &= (1-x)\phi_1(x), \\ B(x) &= (1-x)B_1(x), \\ C(x) &= (1-x)C_1(x), \end{aligned}$$

where

$$\begin{aligned} \phi_1(x) &= (1+x)(1-x^2)^2, \\ B_1(x) &= (1+x^2)(1+x)\big[\beta - \alpha - (\alpha+\beta)x\big], \\ C_1(x) &= \frac{C(x)}{(1-x)}. \end{aligned}$$

Notice that $C_1(-1) \neq 0$. Therefore, it is just to study the behavior of $C_1(x)$ at $x = 1$. Then, applying l'Hôpital rule $C_1(1) = -C'(1)$. So

$$C_1(1) = 8(\alpha+1)M_1.$$

From this equation, two different situations can be considered:

B.1 If $M_1 \neq 0$ then the class of the functional is

$$\tilde{s} = \max\{\deg \phi_1 - 2, \deg \psi_1 - 1\} = 3.$$

B.2 If $M_1 = 0$ then divide by $1 - x$. We denote

$$\begin{aligned} \phi_2(x) &= (1+x)^2(1-x^2), \\ B_2(x) &= (1+x)^2\bigl[\beta-\alpha-(\alpha+\beta)x\bigr], \\ C_2(x) &= \frac{C(x)}{(1-x)^2}. \end{aligned}$$

Notice that in this case

$$\begin{aligned} C(x) &= -\bigl(1-x^2\bigr)^2(\alpha+\beta+1)u_0 \\ &\quad +\bigl(\beta-\alpha-(\alpha+\beta)x\bigr)\bigl[M_2(x^2-1)(x-1)-M_4(x-1)^2\bigr] \\ &\quad +M_2(x^2-1)(x-1)^2+2M_4(x-1)^3, \end{aligned}$$

so

$$\begin{aligned} C_2(x) &= -(1+x)^2(\alpha+\beta+1)u_0 \\ &\quad +\bigl(\beta-\alpha-(\alpha+\beta)x\bigr)\bigl[M_2(1+x)-M_4\bigr]+M_2(x^2-1)-2M_4(1-x). \end{aligned}$$

Evaluating C_2 at $x = -1$,

$$C_2(-1) = -2(\beta+2)M_4 \neq 0.$$

Therefore, the class of the functional is

$$\tilde{s} = \max\{\deg\phi_2 - 2, \deg\psi_2 - 1\} = 2.$$

On the other hand, $x = 1$ will be a zero of the coefficient of $\widetilde{\mathcal{S}}$ if $\alpha = 0$. In this situation,

$$C_2(1) = -4(\beta+1)u_0 \neq 0.$$

Then, the equation cannot be reduced and the class continues to be equal 2.

C If $M_3 \neq 0$ and $M_4 = 0$ the equation can be reduced since the polynomials have $x = -1$ as a common zero. Therefore, the equation can be divided by $1 + x$. We denote

$$\begin{aligned} \phi(x) &= (1+x)\phi_1(x), \\ B(x) &= (1+x)B_1(x), \\ C(x) &= (1+x)C_1(x), \end{aligned}$$

where

$$\begin{aligned} \phi_1(x) &= (1-x)(1-x^2)^2, \\ B_1(x) &= (1-x)(1-x^2)\bigl[\beta-\alpha-(\alpha+\beta)x\bigr], \\ C_1(x) &= \frac{C(x)}{(1+x)}. \end{aligned}$$

Notice that $C_1(1) \neq 0$. Then, we study only $C_1(x)$ at $x = -1$. Applying l'Hôpital rule we get $C_1(-1) = C'(-1)$. So

$$C_1(-1) = 8(\beta+1)M_2,$$

From this equation, two different situations appear

C.1 If $M_2 \neq 0$ then the class of the functional is

$$\tilde{s} = \max\{\deg \phi_1 - 2, \deg \psi_1 - 1\} = 3.$$

C.2 If $M_2 = 0$ then the coefficients of the equation can be divided by $1 + x$. We denote

$$\begin{aligned} \phi_2(x) &= (1-x)^2(1-x^2), \\ B_2(x) &= (1-x)^2\left[\beta - \alpha - (\alpha+\beta)x\right], \\ C_2(x) &= \frac{C(x)}{(1+x)^2}. \end{aligned}$$

Notice that

$$\begin{aligned} C(x) = & -(1-x^2)^2(\alpha+\beta+1)u_0 \\ & +\big(\beta-\alpha-(\alpha+\beta)x\big)\left[M_1(x^2-1)(x+1) - M_3(x+1)^2\right] \\ & -(x^2-1)(x+1)^2 M_1 + 2M_3(x+1)^3, \end{aligned}$$

so

$$\begin{aligned} C_2(x) = & -(1-x)^2(\alpha+\beta+1)u_0 \\ & +\big(\beta-\alpha-(\alpha+\beta)x\big)\left[M_1(x-1) - M_3\right] - M_1(x^2-1) + 2M_3(1+x). \end{aligned}$$

Evaluating C_2 at $x = 1$

$$C_2(1) = 2(\alpha+2)M_3 \neq 0.$$

So, the class is

$$\tilde{s} = \max\{\deg \phi_2 - 2, \deg \psi_2 - 1\} = 2.$$

But, $x = -1$ will be a zero of the coefficient of $\widetilde{\mathcal{S}}$ if $\beta = 0$. In this situation,

$$C_2(-1) = -4(\alpha+1)u_0 \neq 0.$$

Then, the equation cannot be reduced and the class is 2.

As a conclusion, we get

Theorem 19. *Let the moment linear functional, $\tilde{J}_{\alpha,\beta}$, be defined as in (22). The following statements hold*

1. *If $M_3 M_4 \neq 0$ then $\tilde{s} = 4$.*
2. *If $M_3 = 0$ and $M_1 M_4 \neq 0$ then $\tilde{s} = 3$.*
3. *If $M_4 = 0$ and $M_2 M_3 \neq 0$ then $\tilde{s} = 3$.*
4. *If $M_1 = M_3 = 0$ and $M_4 \neq 0$ then $\tilde{s} = 2$.*
5. *If $M_2 = M_4 = 0$ and $M_3 \neq 0$ then $\tilde{s} = 2$.*

Notice that the constants M_3 and M_4 never vanish simultaneously; otherwise the functional $\widetilde{\mathcal{J}}_{\alpha,\beta}$ is reduced to the case introduced in [8].

4.2. **Modified Laguerre functional.** The Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ for $\alpha \in (-1,\infty)$ are defined by the orthogonality and normalization conditions

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)dx = \Gamma(\alpha+1)\left(\begin{array}{c} n+\alpha \\ n \end{array}\right)\delta_{n,m}, \quad n,m\geq 0.$$

These polynomials are orthogonal with respect to the classical Laguerre linear functional $\mathcal{L}_\alpha$. Let consider the following perturbation of such a functional

$$\widetilde{\mathcal{L}}_\alpha = \mathcal{L}_\alpha + M_0\delta(x) + M_1\delta'(x) + \cdots + M_k\delta^{(k)}(x), \quad k\geq 0.$$

The moment functional $\mathcal{L}_\alpha$ satisfies the Pearson equation

$$D\left[x\mathcal{L}_\alpha\right] = (-x+\alpha+1)\mathcal{L}_\alpha.$$

Let

$$\phi(x) = x, \quad \psi(x) = -x+\alpha+1.$$

As we have seen in section 3, from theorem 16, the Stieltjes function associated with the classical Laguerre functional satisfies the differential equation

$$\phi(x)\mathcal{S}'(x) = B(x)\mathcal{S}(x) + C(x), \tag{24}$$

where

$$\begin{aligned} B(x) &= -\phi'(x)+\psi(x) = -x+\alpha, \\ C(x) &= -(U\theta_0\phi)'(x) + (U\theta_0\psi)(x) = -u_0. \end{aligned}$$

Then, the differential equation is

$$x\mathcal{S}'(x) = (-x+\alpha)\mathcal{S}(x) - u_0.$$

Notice that the following relation between both moment functionals hold

$$x^{k+1}\widetilde{\mathcal{L}}_\alpha = x^{k+1}\mathcal{L}_\alpha, \quad k\geq 0.$$

Furthermore,

$$D\left[x^{k+1}\widetilde{\mathcal{L}}_\alpha\right] = (-x+\alpha+k+1)x^k\mathcal{L}_\alpha.$$

This yields the Pearson equation which is satisfied by the perturbed Laguerre linear functional $\widetilde{\mathcal{L}}_\alpha$,

$$\begin{aligned} D\left[x^{k+2}\widetilde{\mathcal{L}}_\alpha\right] &= D\left[xx^{k+1}\widetilde{\mathcal{L}}_\alpha\right] = x^{k+1}\widetilde{\mathcal{L}}_\alpha + xD\left[x^{k+1}\widetilde{\mathcal{L}}_\alpha\right] \\ &= x^{k+1}\widetilde{\mathcal{L}}_\alpha + x(-x+\alpha+k+1)x^k\mathcal{L}_\alpha \\ &= x^{k+1}\widetilde{\mathcal{L}}_\alpha + (-x+\alpha+k+1)x^{k+1}\widetilde{\mathcal{L}}_\alpha. \end{aligned}$$

Therefore,

$$D\left[x^{k+2}\widetilde{\mathcal{L}}_\alpha\right] = (-x+\alpha+k+2)x^{k+1}\widetilde{\mathcal{L}}_\alpha. \tag{25}$$

We assume that $M_k\neq 0$ since in other case the Pearson equation for $\widetilde{\mathcal{L}}_\alpha$ can be simplified.

Now, let us express the Stieltjes function $\widetilde{\mathcal{S}}$ of $\widetilde{\mathcal{L}}_\alpha$ in terms of the Stieltjes function $\mathcal{S}$ associated with $\mathcal{L}_\alpha$. Since

$$\tilde{u}_n = u_n + \sum_{i=0}^{k} M_i u_n^{(i)},$$

and taking into account the Dirac functionals are supported at the origin we get

$$u_n^{(i)} = \begin{cases} (-1)^n n!, & n = i, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \widetilde{\mathcal{S}}(x) &= -\sum_{n=0}^{\infty} \frac{\tilde{u}_n}{x^{n+1}} = -\sum_{n=0}^{\infty} \frac{u_n + M_n u_n^{(n)}}{x^{n+1}} \\ &= \mathcal{S}(x) - \sum_{n=0}^{k} \frac{M_n (-1)^n n!}{x^{n+1}}. \end{aligned}$$

Taking derivatives

$$\widetilde{\mathcal{S}}'(x) = \mathcal{S}'(x) + \sum_{n=0}^{k} \frac{M_n (-1)^n (n+1)!}{x^{n+2}}.$$

Substituting in the above differential equation

$$x\left[\widetilde{\mathcal{S}}'(x) - \sum_{n=0}^{k} \frac{M_n (-1)^n (n+1)!}{x^{n+2}}\right] = (-x+\alpha)\left[\widetilde{\mathcal{S}}(x) - \sum_{n=0}^{k} \frac{M_n (-1)^{n+1} n!}{x^{n+1}}\right] - u_0,$$

so

$$\begin{aligned} x\widetilde{\mathcal{S}}'(x) &= (-x+\alpha)\widetilde{\mathcal{S}}(x) + x\sum_{n=0}^{k} \frac{M_n (-1)^n (n+1)!}{x^{n+2}} - (-x+\alpha)\sum_{n=0}^{k} \frac{M_n (-1)^{n+1} n!}{x^{n+1}} - u_0 \\ &= (-x+\alpha)\widetilde{\mathcal{S}}(x) + \sum_{n=0}^{k} \frac{M_n (-1)^n (n+1)!}{x^{n+1}} \\ &\quad + \sum_{n=0}^{k} \frac{M_n (-1)^{n+1} n!}{x^{n}} - \alpha\sum_{n=0}^{k} \frac{M_n (-1)^{n+1} n!}{x^{n+1}} - u_0. \end{aligned}$$

Taking into account

$$\sum_{n=0}^{k} \frac{M_n (-1)^n (n+1)!}{x^{n+1}} + \sum_{n=0}^{k} \frac{M_n (-1)^{n+1} n!}{x^{n}} + \alpha\sum_{n=0}^{k} \frac{M_n (-1)^n n!}{x^{n+1}}$$

$$= \sum_{n=0}^{k} \frac{M_n (-1)^n n!}{x^{n+1}}(-x+n+1+\alpha) = \frac{\sum_{n=0}^{k} M_n (-1)^n n! (-x+\alpha+n+1) x^{k-n}}{x^{k+1}},$$

we get

$$x\widetilde{\mathcal{S}}'(x) = (-x+\alpha)\widetilde{\mathcal{S}}(x) + \frac{\sum_{n=0}^{k} M_n(-1)^n n!(-x+\alpha+n+1)x^{k-n}}{x^{k+1}} - u_0.$$

Thus,

$$x^{k+2}\widetilde{\mathcal{S}}'(x) = (-x+\alpha)x^{k+1}\widetilde{\mathcal{S}}(x) + \sum_{n=0}^{k} M_n(-1)^n n!(-x+\alpha+n+1)x^{k-n} - u_0 x^{k+1}.$$

In order to reduce the equation we need

$$M_k(-1)^k k!(\alpha+k+1) = 0$$

but, we have already assumed that $M_k \neq 0$, then it cannot be reduced. Therefore, it follows that the order of the class is $k+1$ independently of the values M_1, M_2,..., M_{k-1}.

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